Quantum Morphisms

Lecture 3

Last Week

Main Theorem: If $G \xrightarrow{q} H$, then there exists a perfect quantum strategy for the (G,H)-hom game of the following form:

1) $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$
2) $E_{gh} \in \mathbb{C}^{d \times d}$ & $F_{gh} \in \mathbb{C}^{d \times d}$ are projections $\forall g \in V(G), h \in V(H)$
3) $F_{gh} = E_{gh}^T$ $\qquad g = g' \Rightarrow h = h'$ Synchronous

Main Corollary: $G \xrightarrow{q} H$ if and only if there exists $d \in \mathbb{N}$ & projections $E_{gh} \in \mathbb{C}^{d \times d}$ for all $g \in V(G)$, $h \in V(H)$ satisfying

(1) $\sum_{h \in V(H)} E_{gh} = I$ for all $g \in V(G)$

(2) $E_{gh} E_{g'h'} = 0$ if $(g \sim g'$ & $h \not\sim h')$ or $\underbrace{(g = g' \text{ \& } h \neq h')}_{\text{redundant}}$

$d=1$ case: By Condition (1) there is $\varphi: V(G) \to V(H)$ such that $E_{gh} = \begin{cases} 1 & \text{if } \varphi(g) = h \\ 0 & \text{o.w.} \end{cases}$

$E_{g\varphi(g)} E_{g'\varphi(g')} = 1 \Rightarrow \varphi$ is a homomorphism by Condition (2)

Converse: If $\varphi : V(G) \to V(H)$ is a homomorphism, then setting

$$E_{gh} = \begin{cases} I & \text{if } \varphi(g) = h \\ 0 & \text{o.w.} \end{cases}$$

gives a quantum homomorphism.

Exercise: if the E_{gh} pairwise commute, then $G \to H$.

Transitivity

E_{gh} F_{hk}

Suppose that $G \xrightarrow{q} H$ and $H \xrightarrow{q} K$.

Can we show that $G \xrightarrow{q} K$?

How could they play the (G,K)-hom game?

Idea: play (G,H) game + then (H,K) game.

A + B sent $g, g' \in V(G)$ respectively, obtain $h, h' \in V(H)$ from (G,H) strategy. Then act as if h, h' were inputs to their (H,K) strategy to obtain $k, k' \in V(K)$.

$$g = g' \Rightarrow h = h' \Rightarrow k = k'$$

$$g \sim g' \Rightarrow h \sim h' \Rightarrow k \sim k'$$

In terms of projections: $P_{gk} := \sum_{h \in V(H)} E_{gh} \otimes F_{hk}$

matrix multiplication

$$\sum_k P_{gk} = \sum_{k,h} E_{gh} \otimes F_{hk} = \sum_h E_{gh} \otimes \Big(\underbrace{\sum_k F_{hk}}\Big) = \underbrace{\sum_h E_{gh}}_{=I} \otimes I = I \otimes I = I$$

$g \sim g'$ & $k \not\sim k'$

$$P_{gk} P_{g'k'} = \sum_{h,h'} \underbrace{E_{gh} E_{g'h'}}_{=0 \text{ if } h \not\sim h'} \otimes \underbrace{F_{hk} F_{h'k'}}_{=0 \text{ if } h \sim h'} = 0$$

Quantum Colorings

$G \overset{q}{\to} K_n$ iff $\exists$ projections $E_{gi} \in \mathbb{C}^{d \times d}$ satisfying

1) $\sum_{i=1}^{n} E_{gi} = I \quad \forall g \in V(G)$

2) $E_{gi} E_{g'i} = 0$ if $g \sim g'$

Recall: $\chi(G) = \min\{n : G \to K_n\}$

Definition (Quantum chromatic number):

$$\chi_q(G) := \min\{n : G \overset{q}{\to} K_n\}$$

"On the quantum chromatic number of a graph"
Cameron, Montanaro, Newman, Severini, Winter

Similarities between χ & χ_q

1) $G \to H \Rightarrow \chi(G) \le \chi(H)$ By transitivity
$G \xrightarrow{q} H \Rightarrow \chi_q(G) \le \chi_q(H)$

We say χ_q is monotone w/ respect to quantum homomorphism

2) $G \xrightarrow{q} K_1 \Leftrightarrow G \to K_1$ i.e. G is edgeless.
$E_{g1} = I$

3) $G \xrightarrow{q} K_2 \Leftrightarrow G \to K_2$ i.e. G is bipartite.

<u>Proof:</u>

$E_{u1} = E_{u1}(E_{v1} + E_{v2}) = E_{u1}E_{v2} = (E_{u1} + E_{u2})E_{v2} = E_{v2}$

Similarly $E_{u2} = E_{v1}$ $(E_{v1}, E_{v2}) = (E_{u2}, E_{u1})$

<u>Exercise:</u> Show that $G \xrightarrow{NS} K_2$ for all graphs G.

4) $\chi_q(K_n) = n$

<u>Proof:</u> Exercise.

Uniform rank colorings

Suppose that $E_{gi} \in \mathbb{C}^{d \times d}$ give a quantum n-coloring of G.

Define

$$F_{gi} := \bigoplus_{k=0}^{n-1} E_{g(i+k)} = \begin{pmatrix} E_{gi} & & & 0 \\ & E_{g(i+1)} & & \\ & & \ddots & \\ 0 & & & E_{g(i+n-1)} \end{pmatrix}$$

subscripts modulo n

Then the $F_{gi} \in \mathbb{C}^{nd \times nd}$ give a quantum n-coloring of G with $rk(F_{gi}) = d \quad \forall g \in V(G),\ i \in [n]$.

Definition: $\chi_q^r(G) = \min n$ s.t. G has a quantum n-coloring using only rank r projections. (a rank-r quantum n-coloring of G)

Remark: $\chi_q^r(G) \leq \chi_q^1(G) \leq \chi(G) \quad \forall r \in \mathbb{N}$.

Open Problem: $\chi_q^r(G) \leq n \Rightarrow \chi_q^{r+1}(G) \leq n$?

Suppose that $E_{gi} \in \mathbb{C}^{d \times d}$ give a rank-r quantum n-coloring of G.
Note that $d = rn$. Fix $i \in [n]$ and let $E_g = E_{gi} \ \forall g \in V(G)$.
Then $g \sim g' \Rightarrow E_g E_{g'} = 0$.

The map $g \mapsto E_g$ is a d/r-projective representation of G.
Its value is the rational number $\frac{d}{r}$.

Note: A 3/1-representation is not 6/2-representation, but they have the same value.

Definition (Projective rank)

$\xi_f(G) := \inf\{\frac{d}{r} : G \text{ has a } d/r\text{-representation}\}$

Like a fractional quantum chromatic number.

$\xi_f(G) \leq \chi_q(G), \chi_f(G)$

Open Problems: $\xi_f(G) = \inf\left\{\frac{\chi_q(G[K_n])}{n} : n \in \mathbb{N}\right\}$?

$\xi_f(G) = \inf\{\sqrt[n]{\chi_q(G^{*n})} : n \in \mathbb{N}\}$?

Rank-1 quantum colorings

Suppose $E_{gi} \in \mathbb{C}^{n \times n}$ give a rank-1 quantum n-coloring of G.

$E_{gi} = |\psi_{gi}\rangle\langle\psi_{gi}|$ for some $|\psi_{gi}\rangle \in \mathbb{C}^n$

Conditions on the E_{gi} translate to:

(1) $\{|\psi_{gi}\rangle : i \in [n]\}$ is an orthonormal basis $\forall g \in V(G)$

(2) $\langle\psi_{gi}|\psi_{g'i}\rangle = 0$ (i.e. $|\psi_{gi}\rangle \perp |\psi_{g'i}\rangle$) if $g \sim g'$.

Define $U_g := \sum_{i=1}^{n} |\psi_{gi}\rangle\langle i| \quad \forall g \in V(G)$,

i.e. the i^{th} column of U_g is $|\psi_{gi}\rangle$.

(1) U_g is unitary

(2) $(U_g^* U_{g'})_{ii} = 0$ if $g \sim g'$ (i.e. $U_g^* U_{g'}$ has zero diagonal)

Theorem: A rank-1 quantum n-coloring is equivalent to a homomorphism to $\underbrace{\mathrm{Cay}(U(n), \{U \in U(n) : U_{ii} = 0 \ \forall i\})}_{\text{Unitary derangement graph}}$.

For a fixed $i \in [n]$, the map $g \mapsto |\psi_{gi}\rangle$ is an orthogonal representation of G in dimension n. = $n/1$ - representation

Definition (Orthogonal rank)

$\xi(G) := \min\{n : G$ has an orthogonal representation in $\mathbb{C}^n\}$

$\xi_f(G) \le \xi(G) \le \chi_q'(G)$ but neither $\xi(G) \le \chi_q(G)$ nor $\chi_q(G) \le \xi(G)$ holds for all G.

Constructions

Flat orthogonal representations

$|\psi\rangle \in \mathbb{C}^n$ is flat if all of its entries have the same modulus,
i.e. $|\langle i|\psi\rangle|$ does not depend on i.
unit vector: $1/\sqrt{n}$

Suppose that $g \mapsto |\psi_g\rangle \in \mathbb{C}^n$ is a flat orthogonal representation of G.

F - $n \times n$ flat unitary, e.g. $F_{ij} = \omega^{ij}$, ω a primitive n^{th} root of unity

Let

$D_g = \sqrt{n}\,\mathrm{Diag}(|\psi_g\rangle)$ - unitary

$U_g = D_g F$ - unitary

$$(U_g^* U_{g'})_{ii} = (F^* D_g^* D_{g'} F)_{ii} = 0$$

Lemma: F flat unitary & D diagonal

$\Rightarrow F^* D F$ has constant diagonal.

$$\mathrm{Tr}(F^* D_g^* D_{g'} F) = \mathrm{Tr}(D_g^* D_{g'}) = \langle \psi_g | \psi_{g'} \rangle = 0$$

Example: Ω_n - orthogonality graph of the ± 1 vectors in $\mathbb{R}^n$.

Has flat orthagonal representation in dimension n by construction.

Ω_n is $\begin{cases} \text{edgeless if } n \text{ odd} \\ \text{bipartite if } n \equiv 2 \bmod 4 \end{cases}$

$\chi_q(\Omega_{4n}) \leq 4n \quad = 4n$

Frankl + Rödl (1987): $\chi(\Omega_{4n})$ grows exponentially in n.

Godsil + Newman (2008): $\chi_q(\Omega_{4n}) < \chi(\Omega_{4n}) \Leftrightarrow n \geq 3$.

Remarkable: $\chi_q(\Omega_{4n})$ known exactly but not $\chi(\Omega_{4n})$.

The quaternion trick

$r = (r_0, r_1, r_2, r_3)^T \in \mathbb{R}^4$ unit vector

Then
$$\begin{array}{cccc} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}, & \begin{pmatrix} -r_1 \\ r_0 \\ -r_3 \\ r_2 \end{pmatrix}, & \begin{pmatrix} -r_2 \\ r_3 \\ r_0 \\ -r_1 \end{pmatrix}, & \begin{pmatrix} -r_3 \\ -r_2 \\ r_1 \\ r_0 \end{pmatrix} \\ r^0 & r^1 & r^2 & r^3 \end{array}$$
form an orthonormal basis of $\mathbb{R}^4$

$r \perp s \Rightarrow r^i \perp s^i \quad \forall\ i = 0, 1, 2, 3$

Theorem: if G has an orthogonal representation in $\mathbb{R}^4$, then $\chi_q(G) \leq 4$.

What does this have to do with quaternions?

Define $q(r) = r_0 + r_1 i + r_2 j + r_3 k$.

Then
$$q(r^1) = i\,q(r)$$
$$q(r^2) = j\,q(r)$$
$$q(r^3) = k\,q(r)$$

$\langle r, s \rangle = Re\left(q(r)\overline{q(s)}\right)$

$\langle r^1, s^1 \rangle = Re\left(i q(r)\overline{i q(s)}\right) = Re\left(q(r)\overline{q(s)}\right)$

$\langle r^1, r^2 \rangle = Re\left(i q(r)\overline{j q(r)}\right) = Re\left(-k\, q(r)\overline{q(r)}\right) = k$ coefficient of $q(r)\overline{q(r)}$

Can also use octonions:

Theorem: If $\xi_{\mathbb{R}}(G) \leq 8$, then $\chi_q(G) \leq 8$.